J Stat Phys (2010) 138: 645–661
DOI 10.1007/s10955-009-9888-9

Equilibrium Fluctuations for the Totally Asymmetric Zero-Range Process

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Received: 3 August 2009 / Accepted: 18 November 2009 / Published online: 2 December 2009

Abstract We consider the one-dimensional Totally Asymmetric Zero-Range process evolving on $\mathbb{Z}$ and starting from the Geometric product measure ν_ρ. On the hyperbolic time scale the temporal evolution of the limit density fluctuation field is deterministic, in the sense that the limit field at time t is a translation of the initial one. We consider the system in a reference frame moving at this velocity and we show that the limit density fluctuation field does not evolve in time until $N^{4/3}$, which implies the current across a characteristic to vanish on this longer time scale.

Keywords Totally Asymmetric Zero-Range · Equilibrium Fluctuations · Boltzmann-Gibbs Principle · Multi-scale argument

1 Introduction

In this paper, we study the Totally Asymmetric Zero-Range process (TAZRP) in $\mathbb{Z}$. In this process, if particles are present at a site x, then after a mean one exponential time, one of them jumps to $x+1$ at rate 1, independently of the particles at other sites. This is a Markov process $\eta_\cdot$ with space state $\mathbb{N}^{\mathbb{Z}}$, where the configurations are denoted by η, so that for a site x, $\eta(x)$ represents the number of particles at that site. For each density ρ of particles, there exists an invariant measure denoted by ν_ρ, which is translation invariant and is such that $E_{\nu_\rho}[\eta(0)]=\rho$, that is the Geometric product measure introduced below in (2.1).

Since the work of Rezakhanlou in [9], it is known that for the TAZRP the macroscopic particle density profile in the Euler scaling of time N, evolves according to the hyperbolic conservation law $\partial_t\rho(t,u)+\nabla\phi(\rho(t,u))=0$, where $\phi(\rho)=\frac{\rho}{1+\rho}$. Since ϕ is differentiable, last equation can also be written as $\partial_t\rho(t,u)+\phi'(\rho(t,u))\nabla\rho(t,u)=0$ and characteristics of partial differential equations of this type are straight lines with slope $\phi'(\rho)$. This result is a Law of Large Numbers for the empirical measure related to this process starting from a general set of initial measures associated to a profile ρ_0, see [9] for details. If one wants

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to go further and show a Central Limit Theorem (C.L.T.) for the empirical measure starting from the equilibrium state ν_ρ, one has to consider the density fluctuation field as defined below, see (2.2).

It is not difficult to show that under the hydrodynamic time scale, the limit density fluctuation field at time t is just a translation of the initial density field, which is a Gaussian white noise. The translation or velocity of the system is given by $\phi'(\rho) = \frac{1}{(1+\rho)^2}$ which is the characteristic speed. This same phenomenon happens for the Asymmetric Simple Exclusion process (ASEP) on the hyperbolic time scale, starting from the Bernoulli product measure μ_α, which is an invariant state and the velocity of the system there is given by $1-2\alpha$, see [3] and [4] for details. If we consider the particle system moving in a reference frame with this constant velocity, then the limit density fluctuation field does not evolve in time and one is forced to consider the process evolving on a longer time scale. Following the same approach as in [4] we can accomplish the result for the TAZRP, up to the time scale $N^{4/3}$, i.e. in this case the limit field at time t still coincides with the initial field. Using this approach, the main difficulty in proving the C.L.T. for the empirical measure is showing that the Boltzmann-Gibbs Principle holds for this process, which we can handle by generalizing the multi-scale argument done for the ASEP in [4]. This Principle says, roughly speaking, that non-conserved quantities fluctuate in a faster scale than conserved ones, so when averaging in time a local field, what survives in the limit is its projection over the conserved quantities. To prove last result for the TAZRP there are some extra computations due to the large space state, which we can overcome by using the equivalence of ensembles and a Taylor expansion of the instantaneous flux, in order to avoid the correlation terms. In fact, this result should be valid until the time scale $N^{3/2}$ and this is a first step on that direction.

Since up to the time scale $N^{4/3}$, the macroscopic behavior of the system does only depend on the initial state, this implies that the flux or current of particles across a characteristic vanishes on this longer time scale. If one wants to observe non-trivial fluctuations of this current the process should be speeded up on a longer time scale. In fact, it was recently proved by [2] that the variance of the current across a characteristic is of order $t^{2/3}$ and this translates by saying that in fact our result should hold until the time scale $N^{3/2}$. Indeed, this result should hold for more general systems than TAZRP or ASEP (see [4]), but for the case of one-dimensional systems with one conserved quantity and hydrodynamic equation of hyperbolic type, whose flux is a concave function. This is a step towards showing the universality behavior of the scaling exponent for these systems.

This paper is a natural extension of [4] and the multi-scale argument seems to be robust enough to be able to generalize it to other models and to achieve the conjectured sharp time scale $N^{3/2}$, this is subject to future work.

We remark that all the results presented here, also hold for a more general Zero-Range process, namely one could take a Zero-Range dynamics in which the jump rate from x to $x+1$ is given by $g(\eta(x))$, with g nondecreasing and satisfying conditions of Definition 3.1 of Chap. 2 of [7]. We could also consider partial asymmetric jumps, in the sense that a particle jumps from x to $x+1$ at rate $pg(\eta(x))$ and from x to $x-1$ at rate $qg(\eta(x))$, where $p+q=1$, $p \neq 1/2$ and with g as general as above. The results are valid for these more general processes, but in order to keep the presentation simple we state and prove them for the TAZRP.

An outline of the article follows. In the second section, we introduce the notation and state the main results. In the third section, we consider the process evolving on the hyperbolic time scale and we show the C.L.T. for the current over a fixed bond. In the fourth section, we use the same approach as in [4] to prove the C.L.T. for the empirical measure on a longer time scale and the vanishing of the current across a characteristic. The proof of the Boltzmann-Gibbs Principle is postponed to the fifth section.

2 Statement of Results

The one-dimensional Totally Asymmetric Zero-Range process is the Markov process $\eta_\cdot$ with generator $\mathcal{L}$ given on local functions $f:\mathbb{N}^{\mathbb{Z}}\to\mathbb{R}$ by

$$\mathcal{L}f(\eta)=\sum_{x\in\mathbb{Z}}1_{\{\eta(x)\geq 1\}}(f(\eta^{x,x+1})-f(\eta)),$$

where

$$\eta^{x,x+1}(z)=\begin{cases}\eta(z), & \text{if } z\neq x,x+1,\\ \eta(x)-1, & \text{if } z=x,\\ \eta(x+1)+1, & \text{if } z=x+1.\end{cases}$$

In order to keep notation the more general as we can, we denote by $g(\eta(x))$ the function $1_{\{\eta(x)\geq 1\}}$, which denotes the jump rate of a particle leaving the site x.

The description of this process is the following. At each site, one can have any integer number of particles and after an exponential time of rate one, one of the particles at that site, jumps to the right neighboring site, at rate 1. Initially, place the particles according to a Geometric product measure in $\mathbb{N}^{\mathbb{Z}}$ of parameter $\frac{1}{1+\rho}$, denoted by ν_ρ, which is an invariant measure for the process and has marginal given by:

$$\nu_\rho(\eta:\eta(x)=k)=\left(\frac{\rho}{1+\rho}\right)^k\frac{1}{1+\rho}. \tag{2.1}$$

Since the work of Rezakhanlou in [9], it is known that taking the TAZRP in the Euler time scaling and starting from general probability measures associated to a profile ρ_0 (for details we refer the reader to [9]), one gets in the hydrodynamic limit to the hyperbolic conservation law:

$$\partial_t\rho(t,u)+\nabla\phi(\rho(t,u))=0$$

where the flux $\phi(\cdot)$ is given by $\phi(\rho)=E_{\nu_\rho}[g(\eta(0))]=\frac{\rho}{1+\rho}$.

Fix a configuration η and let $\pi^N(\eta,du)$ denote the empirical measure given by

$$\pi^N(\eta,du)=\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta(x)\delta_{\frac{x}{N}}(du)$$

where δ_u denotes the Dirac measure at u and let $\pi_t^N(\eta,du)=\pi^N(\eta_t,du)$.

In order to state the C.L.T. for the empirical measure we need to define a suitable set of test functions. For an integer $k\geq 0$, denote by $\mathcal{H}_k$ the Hilbert space induced by the Schwartz space $\mathcal{S}(\mathbb{R})$ and the scalar product $\langle f,g\rangle_k=\langle f,K_0^k g\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(\mathbb{R})$, $K_0=x^2-\Delta$ and Δ denotes the usual Laplacian. Denote by $\mathcal{H}_{-k}$ the dual of $\mathcal{H}_k$ relatively to the inner product of $L^2(\mathbb{R})$.

Fix ρ and an integer k. Denote the density fluctuation field by $\mathcal{Y}_\cdot^N$, i.e. the linear functional acting on functions $H\in\mathcal{S}(\mathbb{R})$ as

$$\begin{aligned}\mathcal{Y}_t^N(H)&=\sqrt{N}\big(\langle H,\pi_t^N(\eta,du)\rangle-\mathbb{E}_{\nu_\rho}\langle H,\pi_t^N(\eta,du)\rangle\big)\\&=\frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}}H\left(\frac{x}{N}\right)(\eta_t(x)-\rho),\end{aligned} \tag{2.2}$$

where $\langle H, \pi_t^N(\eta, du)\rangle$ denotes the integral of a test function H with respect to the measure $\pi_t^N(\eta, du)$. Denote by $D(\mathbb{R}^+, \mathcal{H}_{-k})$ (resp. $C(\mathbb{R}^+, \mathcal{H}_{-k})$) the space of $\mathcal{H}_{-k}$-valued functions, right continuous with left limits (resp. continuous), endowed with the uniform weak topology, by $\mathcal{Q}_N$ the probability measure on $D(\mathbb{R}^+, \mathcal{H}_{-k})$ induced by $\mathcal{Y}_\cdot^N$ and ν_ρ, by $\mathbb{P}_{\nu_\rho}$ the probability measure on $D(\mathbb{R}^+, \mathbb{N}^{\mathbb{Z}})$ induced by ν_ρ and $\eta_\cdot$ speeded up by N and denote by $\mathbb{E}_{\nu_\rho}$ the expectation with respect to $\mathbb{P}_{\nu_\rho}$. Here we denote by $\eta_\cdot^N$ the process $\eta_\cdot$ speeded up by N, namely $\eta_t^N = \eta_{tN}$.

Following the same arguments as done for the Symmetric Zero-Range process in Chap. 11 of [7], it is not difficult to prove that the limit density field at time t is a simple translation of the initial one, this is stated as a Theorem below.

Theorem 2.1 *Fix an integer $k > 2$. Denote by $\mathcal{Q}$ the probability measure on $C(\mathbb{R}^+, \mathcal{H}_{-k})$ corresponding to a stationary Gaussian process with mean 0 and covariance given by*

$$E_{\mathcal{Q}}[\mathcal{Y}_t(H)\mathcal{Y}_s(G)] = \chi(\rho)\int_{\mathbb{R}} T_{t-s}H(u)G(u)du$$

for every $0 \le s \le t$ and H, G in $\mathcal{H}_k$. Here $\chi(\rho) = \mathrm{Var}(\eta(0), \nu_\rho) = E_{\nu_\rho}[(\eta(0)-\rho)^2]$ and $T_t H(u) = H(u + \phi'(\rho)t)$. Then, $(\mathcal{Q}_N)_{N\ge 1}$ converges weakly to $\mathcal{Q}$.

Last result holds for the TAZRP evolving in any $\mathbb{Z}^d$ with the appropriate changes. The idea of the proof is to show that $(\mathcal{Q}_N)_N$ is a tight sequence, which implies that it has convergent subsequences and then one characterizes the limiting measure. For the later, one analyzes asymptotically the martingale characterization of the density fluctuation field and shows that the limiting measure $\mathcal{Q}$ is supported on fields $\mathcal{Y}_\cdot$, such that for a fixed time t and a test function H

$$\mathcal{Y}_t(H) = \mathcal{Y}_0(T_t H). \tag{2.3}$$

It is not difficult to show that $\mathcal{Y}_0$ is a Gaussian field with covariance given by $E_{\mathcal{Q}}(\mathcal{Y}_0(G)\mathcal{Y}_0(H)) = \chi(\rho)\langle G, H\rangle$. Concluding, in the hydrodynamic time scale, the fluctuations of the limit field are linearly transported from the initial ones.

Now we introduce the current of particles through a fixed bond. For a site x, let $J_{x,x+1}^N(t)$ be the total number of jumps from the site x to $x+1$ during the time interval $[0, tN]$. Formally one can write

$$J_{x,x+1}^N(t) = \sum_{y\ge x+1}\big(\eta_t^N(y) - \eta_0(y)\big).$$

Since the current of particles can be approximated by the difference between the density fluctuation field at time t and at time zero, evaluated on the Heaviside function, an easy consequence of last result is the derivation of the C.L.T. for the current over a fixed bond, see [5] for details.

Theorem 2.2 *Fix $x \in \mathbb{Z}$, $t \ge 0$ and let*

$$Z_t^N = \frac{1}{\sqrt{N}}\big\{J_{x,x+1}^N(t) - \mathbb{E}_{\nu_\rho}[J_{x,x+1}^N(t)]\big\}.$$

Then, under $\mathbb{P}_{\nu_\rho}$

$$\frac{Z_t^N}{\sqrt{\chi(\rho)\phi'(\rho)}} \underset{N\to+\infty}{\to} B_t$$

weakly, where B_t denotes the standard Brownian motion.

As seen above, in the hyperbolic scaling, the limit density fluctuation field at time t is a translation of the initial one and the translation is given by the characteristic speed, see (2.3). Removing from the system this velocity, the limit field does not evolve in time and one is forced to go beyond the hydrodynamic time scale. In order to see how far we can go to observe the same trivial behavior of the limit density field, we consider the process evolving on the time scale $N^{1+\gamma}$, with $\gamma > 0$. For this process we are able to show that up to the time scale $N^{4/3}$ this is indeed the case. For that let $\eta_t^N = \eta_{tN^{1+\gamma}}$ be the TAZRP evolving on the time scale $N^{1+\gamma}$, fix ρ and redefine the density fluctuation field on $H \in \mathcal{S}(\mathbb{R})$ by:

$$\mathcal{Y}_t^{N,\gamma}(H) = \frac{1}{\sqrt{N}} \sum_{x\in\mathbb{Z}} H\left(\frac{x - \phi'(\rho)tN^{1+\gamma}}{N}\right)(\eta_t(x) - \rho). \tag{2.4}$$

As above, let Q_N^γ be the probability measure on $D(\mathbb{R}^+, \mathcal{H}_{-k})$ induced by $\mathcal{Y}_\cdot^{N,\gamma}$ and ν_ρ, let $\mathbb{P}_{\nu_\rho}^{N,\gamma} = \mathbb{P}_{\nu_\rho}^\gamma$ be the probability measure on $D(\mathbb{R}^+, \mathbb{N}^{\mathbb{Z}})$ induced by ν_ρ and $\eta_\cdot$ speeded up by $N^{1+\gamma}$ and denote by $\mathbb{E}_{\nu_\rho}^\gamma$ the expectation with respect to $\mathbb{P}_{\nu_\rho}^\gamma$.

Theorem 2.3 *Fix an integer $k > 1$ and $\gamma < 1/3$. Let $\mathcal{Q}$ be the probability measure on $C(\mathbb{R}^+, \mathcal{H}_{-k})$ corresponding to a stationary Gaussian process with mean* 0 *and covariance given by*

$$E_{\mathcal{Q}}[\mathcal{Y}_t(H)\mathcal{Y}_s(G)] = \chi(\rho) \int_{\mathbb{R}} H(u)G(u)du$$

for every $s, t \geq 0$ and H, G in $\mathcal{H}_k$. Then, $(\mathcal{Q}_N^\gamma)_{N\geq\mathcal{Q}1}$ converges weakly to $\mathcal{Q}$.

The main difficulty to overcome when showing last result is the Boltzmann-Gibbs Principle, which we can prove for $\gamma < 1/3$ by applying a multi-scale argument as done for the ASEP in [4].

Theorem 2.4 (Boltzmann-Gibbs Principle) *Fix $\gamma < 1/3$. For every $t > 0$ and $H \in \mathcal{S}(\mathbb{R})$,*

$$\lim_{N\to\infty} \mathbb{E}_{\nu_\rho}^\gamma \left[\left(\int_0^t \frac{N^\gamma}{\sqrt{N}} \sum_{x\in\mathbb{Z}} H\left(\frac{x}{N}\right) V_g(\eta_s^N(x))ds\right)^2\right] = 0,$$

where

$$V_g(\eta(x)) = g(\eta(x)) - \phi(\rho) - \phi'(\rho)(\eta(x) - \rho) \tag{2.5}$$

and $\phi(\rho) = E_{\nu_\rho}[g(\eta(0))]$.

The special features of the process that one uses to derive this result, when applying Proposition A1.6.1 of [7], are: the spectral gap bound for the corresponding symmetric dynamics on boxes of fixed size and the equivalence of ensembles (see Corollary A2.1.7 of [7]). As mentioned in the introduction this result should be valid for $\gamma < 1/2$, which corresponds to $N^{3/2}$. Since we apply the Proposition mentioned above, which bounds the expectation appearing in the statement of the Boltzmann-Gibbs Principle by the square of the $\mathcal{H}_{-1}$-norm of the function that is inside the time integral and since this norm does not capture the asymmetry of the process, the correct time scale is not achieve for asymmetric systems as the ASEP (see [4]) or the TAZRP. Nevertheless, the multi-scale argument seems robust and sharp enough to obtain the results for symmetric systems, see Corollary 7.4 of [4] and Theorem 1 of [1].

Since in this longer time scale the process is moving in a reference frame we define now the current of particles across a characteristic. Let $J^{N,\gamma}_{v^x_t}(t)$ be the current through the moving bond $[v^x_t, v^x_t+1]$ (where $v^x_t = x + [\phi'(\rho)tN^{1+\gamma}]$) defined as the number of particles that jump from v^x_t to v^x_t+1, from time 0 to $tN^{1+\gamma}$:

$$J^{N,\gamma}_{v^x_t}(t) = \sum_{y\geq 1}\big(\eta^N_t(y+v^x_t) - \eta_0(y+x)\big).$$

Since up to this longer time scale the limit density field does not evolve in time and since this current can be approximated by the difference between the density field at time t and at time 0 evaluated on the Heaviside function, it holds that:

Proposition 2.5 *Fix $t \geq 0$, a site $x \in \mathbb{Z}$ and $\gamma < 1/3$. Then,*

$$\lim_{N\to+\infty} \mathbb{E}^{\gamma}_{\nu_\rho}\bigg[\bigg(\frac{1}{\sqrt{N}}\big\{J^{N,\gamma}_{v^x_t}(t) - \mathbb{E}^{\gamma}_{\nu_\rho}\big[J^{N,\gamma}_{v^x_t}(t)\big]\big\}\bigg)^2\bigg] = 0.$$

3 The Hyperbolic Scaling

Recall that as argued above the density fluctuation field $\mathcal{Y}^N_t$ converges to the field $\mathcal{Y}_t$ that depends only on the initial density field $\mathcal{Y}_0$, which is a Gaussian white noise, see (2.3).

Now we give a sketch of the proof of Theorem 2.2, namely we establish the C.L.T. for the current over a fixed bond $[x, x+1]$. For simplicity and since the invariant measure is homogeneous, we prove the result for the bond $[-1, 0]$, but for any other bond the same argument applies. The idea of the proof is to show the convergence of the finite dimensional distributions of $Z^N_t/\sqrt{\chi(\rho)\phi'(\rho)}$ to those of Brownian motion, together with tightness.

We start by the former, namely, first we prove that for every $k \geq 1$ and every $0 \leq t_1 < \cdots < t_k$, $(Z^N_{t_1}, \ldots, Z^N_{t_k})$ converges in law to a Gaussian vector $(Z_{t_1}, \ldots, Z_{t_k})$, with mean zero and covariance given by $E_{\mathcal{Q}}[Z_t Z_s] = \chi(\rho)\phi'(\rho)s$, provided $s \leq t$. We notice here that the covariance of Z_t and Z_s does only depends on the smaller time s, by the fact that the current can be written in terms of the density fluctuation field together with the fact that the fluctuations of this field are linearly transported in time. In order to establish this first claim, notice that

$$J^N_{-1,0}(t) = \sum_{x\geq 0}\big(\eta^N_t(x) - \eta_0(x)\big),$$

then, formally it holds that

$$\frac{1}{\sqrt{N}}\big\{J^N_{-1,0}(t) - \mathbb{E}_{\nu_\rho}[J^N_{-1,0}(t)]\big\} = \mathcal{Y}^N_t(H_0) - \mathcal{Y}^N_0(H_0),$$

where H_0 is the Heaviside function defined as $H_0(u) = 1_{[0,\infty)}(u)$. Approximating H_0 by $(G_n)_{n\geq 1}$, such that $G_n(u) = (1-\frac{u}{n})^+ 1_{[0,\infty)}(u)$, then we can show that

Proposition 3.1 *For every $t \geq 0$,*

$$\lim_{n\to+\infty} \mathbb{E}_{\nu_\rho}\bigg[\bigg(\frac{1}{\sqrt{N}}\big\{J^N_{-1,0}(t) - \mathbb{E}_{\nu_\rho}[J^N_{-1,0}(t)]\big\} - (\mathcal{Y}^N_t(G_n) - \mathcal{Y}^N_0(G_n))\bigg)^2\bigg] = 0$$

uniformly in N.

Proof Fix a site x, use the martingale representation of the current $J^N_{x,x+1}(t)$ as:

$$M^N_{x,x+1}(t) = J^N_{x,x+1}(t) - N\int_0^t g(\eta^N_s(x))ds.$$

This is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(\eta^N_s, s \leq t)$, whose quadratic variation is given by

$$\langle M^N_{x,x+1}\rangle_t = N\int_0^t g(\eta^N_s(x))ds.$$

Since $J^N_{x-1,x}(t) - J^N_{x,x+1}(t) = \eta^N_t(x) - \eta_0(x)$ for all $x \in \mathbb{Z}$ and $t \geq 0$, it holds that

$$\mathcal{Y}^N_t(G_n) - \mathcal{Y}^N_0(G_n) = \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}} G_n\left(\frac{x}{N}\right)\left\{J^N_{x-1,x}(t) - J^N_{x,x+1}(t)\right\}.$$

Introducing the expectation of the current inside the brackets on the right hand side of last expression and making a summation by parts, by using the explicit knowledge of G_n one can write the expectation in the statement of the Proposition as

$$\mathbb{E}_{\nu_\rho}\left[\left(\frac{1}{\sqrt{N}}\sum_{x=1}^{Nn}\frac{1}{Nn}\left\{J^N_{x-1,x}(t) - \mathbb{E}_{\nu_\rho}[J^N_{x-1,x}(t)]\right\}\right)^2\right].$$

Using again the martingale representation of the current $J^N_{x-1,x}(t)$, last expression becomes equal to

$$\mathbb{E}_{\nu_\rho}\left[\left(\frac{1}{\sqrt{N}}\sum_{x=1}^{Nn}\frac{1}{Nn}M^N_{x-1,x}(t) + \frac{1}{\sqrt{N}}\int_0^t\frac{1}{n}\sum_{x=1}^{Nn}(g(\eta_s(x-1)) - \phi(\rho))ds\right)^2\right].$$

Now the goal consists in showing that this expectation vanishes as $n \to +\infty$ uniformly over N. For that, notice that the martingale term converges to 0 in $L^2(\mathbb{P}_{\nu_\rho})$ as $n \to +\infty$, since one can estimate their quadratic variation by Nt and using the fact that they are orthogonal, to obtain that

$$\mathbb{E}_{\nu_\rho}\left[\left(\frac{1}{\sqrt{N}}\sum_{x=1}^{Nn}\frac{1}{Nn}M^N_{x-1,x}(t)\right)^2\right] \leq \frac{tC}{Nn}.$$

On the other hand, one can use Schwarz inequality to get to

$$\mathbb{E}_{\nu_\rho}\left[\left(\frac{1}{\sqrt{N}}\int_0^t\frac{1}{n}\sum_{x=1}^{Nn}(g(\eta_s(x-1)) - \phi(\rho))ds\right)^2\right] \leq \frac{t^2\,\mathrm{Var}(g,\nu_\rho)}{n}.$$

Since $(x+y)^2 \leq 2x^2 + 2y^2$ and taking the limit as $n \to \infty$ in the previous expectations, our proof is concluded. □

The convergence of finite dimensional distributions is an easy consequence of last result together with Theorem 2.1, for details we refer the reader to [5].

Now, it remains to prove that the distributions of $Z^N_t/\sqrt{\chi(\rho)\phi'(\rho)}$ are tight. For that, we can use the same argument as in Theorem 2.3 of [4], that relies on the use of Theorem 2.1

of [10] with the definition of weakly positive associated increments given in [11]. One can follow the same arguments as those of Theorem 2 of [6] to show that the flux of particles through the bond $[-1,0]$ from time 0 to time t, denoted by $J_{-1,0}(t)$ has weakly positive associated increments with the definition in [11], see [4] for details. In order to conclude the proof it remains to note that

$$\lim_{t\to+\infty}\frac{1}{t}E_{\nu_\rho}\big[\big(J_{-1,0}(t)\big)^2\big]=\sigma^2,$$

which follows by Theorem 3 of [6].

4 The Longer Time Scale

4.1 Equilibrium Fluctuations on the Longer Time Scale

In this section we want to prove Theorem 2.3. For that, fix a positive integer k, let $U_t^N H(u)=H(u-\phi'(\rho)tN^\gamma)$ and recall the definition of $(Q_N^\gamma)_{N\geq 1}$. We want to show that this sequence is tight and to characterize the limiting measure. Notice that following the same computations as done for the ASEP in Sect. 8 of [4], it is easy to show that the sequence $(Q_N^\gamma)_{N\geq 1}$ is tight. We leave this computation to the reader. From this result, we impose the condition $k>1$, in order to have the field $\mathcal{Y}_t$ well defined on the Sobolev space $\mathcal{H}_k$.

Now we characterize the limit field, by fixing $H\in\mathcal{S}(\mathbb{R})$ such that

$$\begin{aligned}M_t^{N,\gamma}(H)=\mathcal{Y}_t^{N,\gamma}(H)-\mathcal{Y}_0^{N,\gamma}(H)\\&-\int_0^t\bigg(\frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\nabla^N U_s^N H\Big(\frac{x}{N}\Big)g(\eta_s^N(x))\\&-\frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\partial_u U_s^N H\Big(\frac{x}{N}\Big)\phi'(\rho)(\eta_s^N(x)-\rho)\bigg)ds\end{aligned}$$

is a martingale with respect to the natural filtration $\sigma(\eta_s^N,s\leq t)$ and whose quadratic variation is given by

$$\int_0^t\frac{N^\gamma}{N^2}\sum_{x\in\mathbb{Z}}\Big(\nabla^N U_s^N H\Big(\frac{x}{N}\Big)\Big)^2 g(\eta_s^N(x))ds.$$

If $\gamma<1$, $M_t^{N,\gamma}(H)$ vanishes in $L^2(\mathbb{P}_{\nu_\rho}^\gamma)$, as $N\to+\infty$. This means that under the diffusive time scale there is only a contribution to the limit density field given by the integral part of the martingale.

Now, we want to show that the integral part of the martingale $M_t^{N,\gamma}(H)$ vanishes in $L^2(\mathbb{P}_{\nu_\rho}^\gamma)$, as $N\to+\infty$. For this, we can use the fact that $\sum_{x\in\mathbb{Z}}\nabla^N U_s^N H(\frac{x}{N})=0$ to introduce it times $\phi(\rho)$ in the integral part of the martingale $M_t^{N,\gamma}(H)$ and write it as:

$$\begin{aligned}&\int_0^t\bigg(\frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\nabla^N U_s^N H\Big(\frac{x}{N}\Big)(g(\eta_s^N(x))-\phi(\rho))\\&-\frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\partial_u U_s^N H\Big(\frac{x}{N}\Big)\phi'(\rho)(\eta_s^N(x)-\rho)\bigg)ds.\end{aligned}$$

By summing and subtracting

$$\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\nabla^N U_s^N H\Big(\frac{x}{N}\Big)\phi'(\rho)(\eta_s^N(x)-\rho)ds,$$

to the expression above, one can write the integral part of the martingale as:

$$\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\nabla^N U_s^N H\Big(\frac{x}{N}\Big)V_g(\eta_s^N(x)) \\ -\frac{N^\gamma}{\sqrt{N}}\sum_{x\in\mathbb{Z}}\Big(\partial_u U_s^N H\Big(\frac{x}{N}\Big)-\nabla^N U_s^N H\Big(\frac{x}{N}\Big)\Big)\phi'(\rho)(\eta_s^N(x)-\rho)ds,$$

where $V_g(\eta(x))$ is defined in (2.5).

Using Schwarz inequality, the fact that ν_ρ is a product invariant measure and a Taylor expansion on $U_s^N H$, last integral vanishes as $N\to+\infty$, as long as $\gamma<1$.

Using the Boltzmann-Gibbs Principle, whose proof is sketched in the next section, the first integral in the expression above vanishes in $L^2(\mathbb{P}^\gamma_{\nu_\rho})$ as $N\to+\infty$ if $\gamma<1/3$. This in turn implies that if $\mathcal{Q}$ is one limiting point of $(\mathcal{Q}_N)_N$, then it is supported on a field $\mathcal{Y}_\cdot$ that satisfies $\mathcal{Y}_t(H)=\mathcal{Y}_0(H)$ for $H\in\mathcal{S}(\mathbb{R})$ and $\mathcal{Y}_0$ is a Gaussian field with covariance given by $E_{\mathcal{Q}}(\mathcal{Y}_0(G)\mathcal{Y}_0(H))=\chi(\rho)\langle G,H\rangle$. This concludes the proof of Theorem 2.3.

Remark 4.1 We remark here that as mentioned above, the Boltzmann-Gibbs Principle should hold for $\gamma<1/2$, which implies that until the time scale $N^{3/2}$ the temporal evolution of the density field is trivial. Once the Boltzmann-Gibbs Principle is proved to hold for this longer time scale, then following the same arguments as above, one obtains the result of Theorem 2.3 for the longer time scale $N^{3/2}$.

4.2 Current Across a Characteristic

Here we want to prove Proposition 2.5. As in the hyperbolic scaling, it is a consequence of next result together with Theorem 2.3. For details we refer the reader to [5].

Proposition 4.1 *For every $t\geq 0$ and $\gamma<1/3$:*

$$\lim_{n\to+\infty}\mathbb{E}^\gamma_{\nu_\rho}\Big[\Big(\frac{1}{\sqrt{N}}\big\{J^{N,\gamma}_{v_t^x}(t)-\mathbb{E}^\gamma_{\nu_\rho}\big[J^{N,\gamma}_{v_t^x}(t)\big]\big\}-\big(\mathcal{Y}^{N,\gamma}_t(G_n)-\mathcal{Y}^{N,\gamma}_0(G_n)\big)\Big)^2\Big]=0,$$

uniformly over N.

The proof of this result follows the same lines as the proof of Proposition 9.4 in [4] and for that reason we have omitted it. This result implies that in order to observe non-trivial fluctuations of this current the process has to be taken on a longer time scale than $N^{4/3}$. In fact, we remark here that once the Boltzmann-Gibbs Principle as stated above is proven for $\gamma<1/2$, then applying the same argument one can show that the current across a characteristic also vanishes up to the longer time scale $N^{3/2}$.

5 The Boltzmann-Gibbs Principle

In this section we prove Theorem 2.4. Since we are going to generalize the ideas in Theorem 2.6 of [4] we just remark the novelty and the fundamental differences between the proofs.

Fix an integer K and a test function $H \in \mathcal{S}(\mathbb{R})$ and divide $\mathbb{Z}$ in non overlapping intervals of length K, denoted by $\{I_j, j \geq 1\}$.

Then, the expectation appearing in the statement of the Theorem can be written as

$$\mathbb{E}^{\gamma}_{\nu_\rho}\bigg[\bigg(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\geq 1}\sum_{x\in I_j} H\Big(\frac{x}{N}\Big)V_g(\eta^N_s(x))ds\bigg)^2\bigg].$$

At this stage, the first argument is to replace the empirical mean of H in each interval I_j by the value of H in y_j/N, where y_j is a certain point of that interval. For this, we use Schwarz inequality and the price to pay for this replacement is given by a restriction on the size K of the interval. The second argument is to replace the empirical mean of V_g in each interval I_j, by its projection over the conserved quantities of the corresponding interval I_j. This also brings us a restriction on the size K of the interval I_j. Combining both, we can take the biggest interval in which these two substitutions can take place. We condition the function V_g on the locally conserved quantities, since as mentioned in the introduction, they evolve in a much slower scale than non-conserved ones and the projection of the non-conserved quantities over the conserved ones is what survives in the limit. We start by computing explicitly, the restrictions on K in order to perform these two first replacements. Then, the proof follows by applying these two arguments to intervals of bigger size until a point in which the remaining is negligible.

With this in mind, for each $j \geq 1$ we start by fixing a point y_j of the interval I_j. Summing and subtracting $H(\frac{y_j}{N})$ inside the summation over x, and since $(x+y)^2 \leq 2x^2+2y^2$, we can bound the expectation above by

$$2\mathbb{E}^{\gamma}_{\nu_\rho}\bigg[\bigg(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\geq 1}\sum_{x\in I_j}\Big(H\Big(\frac{x}{N}\Big)-H\Big(\frac{y_j}{N}\Big)\Big)V_g(\eta^N_s(x))ds\bigg)^2\bigg]$$
$$+\,2\mathbb{E}^{\gamma}_{\nu_\rho}\bigg[\bigg(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\geq 1} H\Big(\frac{y_j}{N}\Big)\sum_{x\in I_j}V_g(\eta^N_s(x))ds\bigg)^2\bigg].$$

Now, we treat each expectation separately. For the former, the idea is to replace the empirical mean of H in the interval I_j, by the value of $H(\frac{y_j}{N})$. Notice that this expectation can also be written as

$$2\mathbb{E}^{\gamma}_{\nu_\rho}\bigg[\bigg(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\geq 1}K\Big(\frac{1}{K}\sum_{x\in I_j}H\Big(\frac{x}{N}\Big)-H\Big(\frac{y_j}{N}\Big)\Big)V_g\big(\eta^N_s(x)\big)ds\bigg)^2\bigg],$$

which is easily handled, since by Schwarz inequality and the invariance of ν_ρ it can be bounded by $Ct^2N^{2\gamma}\|H'\|_2^2(\frac{K}{N})^2$ and vanishes as long as $KN^{\gamma-1}\to 0$, when $N\to+\infty$.

In order to treat the remaining expectation we bound it from above by

$$2\mathbb{E}^{\gamma}_{\nu_\rho}\bigg[\bigg(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\geq 1} H\Big(\frac{y_j}{N}\Big)V_{1,j,g}(\eta^N_s)ds\bigg)^2\bigg]$$

$$+2\mathbb{E}_{\nu_\rho}^{\gamma}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\geq 1} H\Big(\frac{y_j}{N}\Big) E_{\nu_\rho}\Big[\sum_{x\in I_j} V_g(\eta_s^N(x))\big|M_j\Big]ds\Big)^2\Big] \tag{5.1}$$

where

$$V_{1,j,g}(\eta)=\sum_{x\in I_j} V_g(\eta(x)) - E_{\nu_\rho}\Big[\sum_{x\in I_j} V_g(\eta(x))\big|M_j\Big]$$

and $M_j=\sigma(\sum_{x\in I_j}\eta(x))$. We treat now the first expectation of the expression above. Notice as above, that it can also be written as

$$\mathbb{E}_{\nu_\rho}^{\gamma}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\geq 1} H\Big(\frac{y_j}{N}\Big)K\Big(\frac{1}{K}\sum_{x\in I_j} V_g(\eta_s^N(x)) - E_{\nu_\rho}\big[V_g(\eta_s^N(x))\big|M_j\big]\Big)ds\Big)^2\Big].$$

So, at this stage, we have to replace the empirical mean of V_g in I_j, by its projection over the conserved quantities of the interval I_j. Since we are restricted to sets of size K, the conserved quantities are the set of configurations with a fixed number of particles. The Lemma below, tell us, how big the size K of the set can be, in order to perform this replacement.

Lemma 5.1 *For every $H\in\mathcal{S}(\mathbb{R})$ and every $t>0$, if $K^2N^{\gamma-1}\to 0$ as $N\to+\infty$, then*

$$\lim_{N\to\infty}\mathbb{E}_{\nu_\rho}^{\gamma}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{j\geq 1} H\Big(\frac{y_j}{N}\Big)V_{1,j,g}(\eta_s^N)ds\Big)^2\Big]=0.$$

Proof By Proposition A1.6.1 of [7] and by the variational formula for the $\mathcal{H}_{-1}$-norm, the expectation above is bounded by

$$Ct\sum_{j\geq 1}\sup_{h\in L^2(\nu_\rho)}\Big\{2\int \frac{N^\gamma}{\sqrt{N}}H\Big(\frac{y_j}{N}\Big)V_{1,j,g}(\eta)h(\eta)\nu_\rho(d\eta) - N^{1+\gamma}\langle h, -\mathcal{L}_{I_j}^S h\rangle_\rho\Big\},$$

where $\mathcal{L}^S$ is the Symmetric Zero-Range dynamics restricted to the set I_j, namely:

$$\mathcal{L}_{I_j}^S f(\eta)=\sum_{\substack{x,y\in I_j\\|x-y|=1}}\frac{1}{2}g(\eta(x))[f(\eta^{x,y})-f(\eta)],$$

where

$$\eta^{x,y}(z)=\begin{cases}\eta(z), & \text{if } z\neq x,y,\\ \eta(x)-1, & \text{if } z=x,\\ \eta(y)+1, & \text{if } z=y.\end{cases}$$

For each j and A_j a positive constant, it holds that

$$\int V_{1,j,g}(\eta)h(\eta)\nu_\rho(d\eta)\leq \frac{1}{2A_j}\langle V_{1,j,g},(-\mathcal{L}_{I_j}^S)^{-1}V_{1,j,g}\rangle_\rho+\frac{A_j}{2}\langle h,-\mathcal{L}_{I_j}^S h\rangle_\rho,$$

and taking $A_j=N^{3/2}(|H(\frac{y_j}{N})|)^{-1}$, the whole expectation becomes bounded from above by

$$Ct\sum_{j\geq 1}\frac{N^\gamma}{N^2}H^2\Big(\frac{y_j}{N}\Big)\langle V_{1,j,g},(-\mathcal{L}_{I_j}^S)^{-1}V_{1,j,g}\rangle_\rho.$$

By the spectral gap inequality for the Symmetric Zero-Range process (see [8]) last expression can be bounded from above by

$$Ct\sum_{j\geq 1}\frac{N^\gamma}{N^2}H^2\Big(\frac{y_j}{N}\Big)(K+1)^2\,\mathrm{Var}(V_{1,j,g},\nu_\rho).$$

The proof of the Lemma ends if we show that $\mathrm{Var}(V_{1,j,g},\nu_\rho)\leq KC$, since it implies that the expectation in the statement of the Lemma to be bounded by $Ct\frac{N^\gamma}{N}(K+1)^2\|H\|_2^2$ and vanishes as long as $K^2N^{\gamma-1}\to 0$, when $N\to+\infty$. □

Remark 5.1 Here we show that $\mathrm{Var}(V_{1,j,g},\nu_\rho)\leq KC$.

Since $\mathrm{Var}(V_{1,j,g},\nu_\rho)\leq E_{\nu_\rho}[(V_{1,j,g})^2]$ and by the definition of $V_{1,j,g}$ it holds that

$$\mathrm{Var}(V_{1,j,g},\nu_\rho)\leq E_{\nu_\rho}\Bigg[\bigg(\sum_{x\in I_j}V_g(\eta(x))-E_{\nu_\rho}\Big[\sum_{x\in I_j}V_g(\eta(x))|M_j\Big]\bigg)^2\Bigg].$$

By the definition of $V_g(\eta(x))$, the right hand side of last expression can be written as

$$E_{\nu_\rho}\Bigg[\bigg(\sum_{x\in I_j}\big(g(\eta(x))-\phi(\rho)-\phi'(\rho)(\eta(x)-\rho)\big)-\sum_{x\in I_j}\phi_j(\rho)-K\phi(\rho)$$
$$-\sum_{x\in I_j}\phi'(\rho)(\eta_j^K-\rho)\bigg)^2\Bigg],$$

where $\phi_j(\rho)=E_{\nu_\rho}[g(\eta(0))|M_j]$ and $\eta_j^K=\frac{1}{K}\sum_{x\in I_j}\eta(x)$. On the other hand, by summing and subtracting $\phi(\eta_j^K)=E_{\nu_{\eta_j^K}}[g(\eta)]$, where $\nu_{\eta_j^K}$ is the Bernoulli measure with density η_j^K, last expression can be bounded by

$$4E_{\nu_\rho}\Bigg[\bigg(\sum_{x\in I_j}\big(g(\eta(x))-\phi(\rho)\big)\bigg)^2\Bigg]+4E_{\nu_\rho}\Bigg[\bigg(\sum_{x\in I_j}\phi'(\rho)(\eta(x)-\rho)\bigg)^2\Bigg]$$
$$+4E_{\nu_\rho}\Bigg[\bigg(\sum_{x\in I_j}\big(\phi_j(\rho)-\phi(\eta_j^K)\big)\bigg)^2\Bigg]$$
$$+4E_{\nu_\rho}\Bigg[\bigg(\sum_{x\in I_j}\big(\phi(\eta_j^K)-\phi(\rho)-\phi'(\rho)(\eta_j^K-\rho)\big)\bigg)^2\Bigg].$$

Now we treat each expectation separately.

For the first and the second one, since $(\eta(x))_x$ are independent under ν_ρ, it is easy to show that

$$E_{\nu_\rho}\Bigg[\bigg(\sum_{x\in I_j}\big(g(\eta(x))-\phi(\rho)\big)\bigg)^2\Bigg]\leq K\,\mathrm{Var}(g,\nu_\rho)$$

and

$$E_{\nu_\rho}\Bigg[\bigg(\sum_{x\in I_j}\phi'(\rho)(\eta(x)-\rho)\bigg)^2\Bigg]\leq(\phi'(\rho))^2K\,\mathrm{Var}(\eta(0),\nu_\rho).$$

On the other hand, to treat the third expectation one can use the equivalence of ensembles (see Corollary A2.1.7 of [7]) which guarantees that $|\phi_j(\rho)-\phi(\eta_j^K)|\leq\frac{C(g)}{K}$ while for the last one, one can use Taylor expansion to have

$$E_{\nu_\rho}\big[\big(\phi(\eta_j^K)-\phi(\rho)-\phi'(\rho)(\eta_j^K-\rho)\big)^2\big]\sim E_{\nu_\rho}\big[\big(\eta_j^K-\rho\big)^4\big]=O(K^{-2}).$$

Putting these arguments all together one gets to the bound KC.

We notice here that collecting together the two restrictions on K given above, in order to perform both replacements in sets of size K, one must have $K\sim O(N^{\frac{1-\gamma}{2}-\epsilon})$, for $\epsilon>0$.

To conclude the proof it remains to bound the second expectation on (5.1). This is the second stage of the argument in which we are going to take bigger sets in order to perform two replacements as above. For that, fix an integer L and take disjoint intervals of length $M=LK$, denoted by $\{I_l^2, l\geq 1\}$ and write that expectation as:

$$\mathbb{E}^\gamma_{\nu_\rho}\bigg[\bigg(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{l\geq 1}\sum_{j\in I_l^2}H\Big(\frac{y_j}{N}\Big)E_{\nu_\rho}\Big[\sum_{x\in I_j}V_g(\eta_s^N(x))\Big|M_j\Big]ds\bigg)^2\bigg].$$

At the first step we want to replace the empirical mean of H in each one of the sets of size M (taking K as scale factor) by the value of H in z_l/N, where z_l is a certain point of the interval I_l^2. Then, for each $l\geq 1$, fix a point z_l in I_l^2, by summing and subtracting $H(\frac{z_l}{N})$ inside the summation over j, last expectation can be bounded by

$$2\mathbb{E}^\gamma_{\nu_\rho}\bigg[\bigg(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{l\geq 1}\sum_{j\in I_l^2}\Big(H\Big(\frac{y_j}{N}\Big)-H\Big(\frac{z_l}{N}\Big)\Big)E_{\nu_\rho}\Big[\sum_{x\in I_j}V_g(\eta_s^N(x))\Big|M_j\Big]ds\bigg)^2\bigg]$$
$$+2\mathbb{E}^\gamma_{\nu_\rho}\bigg[\bigg(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{l\geq 1}H\Big(\frac{z_l}{N}\Big)\sum_{j\in I_l^2}E_{\nu_\rho}\Big[\sum_{x\in I_j}V_g(\eta_s^N(x))\Big|M_j\Big]ds\bigg)^2\bigg].$$

Now, notice that the first expectation above can be written as

$$\mathbb{E}^\gamma_{\nu_\rho}\bigg[\bigg(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{l\geq 1}L\Big(\frac{1}{L}\sum_{j\in I_l^2}H\Big(\frac{y_j}{N}\Big)-H\Big(\frac{z_l}{N}\Big)\Big)$$
$$\times E_{\nu_\rho}\Big[\sum_{x\in I_j}V_g(\eta_s^N(x))\Big|M_j\Big]ds\bigg)^2\bigg].$$

Following the same arguments as above it is easy to show that, under the Schwarz inequality, this expectation vanishes if $L^2KN^{2\gamma-2}\to 0$ as $N\to+\infty$. This is the first restriction on the size M of the intervals $(I_l^2)_l$.

In order to treat the second expectation, inside the summation over l sum and subtract $E_{\nu_\rho}[\sum_{x\in I_l^2}V_g(\eta_s^N(x))|M_l^2]$, where $M_l^2=\sigma(\sum_{x\in I_l^2}\eta(x))$ and bound it from above by

$$2\mathbb{E}^\gamma_{\nu_\rho}\bigg[\bigg(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{l\geq 1}H\Big(\frac{z_l}{N}\Big)V_{2,l,g}(\eta_s^N)ds\bigg)^2\bigg]$$
$$+2\mathbb{E}^\gamma_{\nu_\rho}\bigg[\bigg(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{l\geq 1}H\Big(\frac{z_l}{N}\Big)E_{\nu_\rho}\Big[\sum_{x\in I_l^2}V_g(\eta_s^N(x))\Big|M_l^2\Big]ds\bigg)^2\bigg],$$

where

$$V_{2,l,g}(\eta) = \sum_{j\in I_l^2} E_{\nu_\rho}\Big[\sum_{x\in I_j} V_g(\eta(x))\Big| M_j\Big] - E_{\nu_\rho}\Big[\sum_{x\in I_l^2} V_g(\eta(x))\Big| M_l^2\Big].$$

Now we treat the first expectation of last expression. Notice that it can also be written as

$$2\mathbb{E}^{\gamma}_{\nu_\rho}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{l\geq 1} H\Big(\frac{z_l}{N}\Big) M\Big(\frac{1}{L}\sum_{j\in I_l^2} E_{\nu_\rho}\big[V_g(\eta_s^N(x))\big|M_j\big] \\ - E_{\nu_\rho}\big[V_g(\eta_s^N(x))\big|M_l^2\big]\Big)ds\Big)^2\Big].$$

This is the second step at this stage of the argument. Here, we want to replace the empirical mean in the intervals of size L of the projection of V_g on the conserved quantities of the intervals of size K, by its projection on the conserved quantities of the intervals of size M. This will bring us the second restriction on the size M of the intervals $(I_l^2)_l$ that allow us to perform both replacements. As above we prove that:

Lemma 5.2 *For every $H \in \mathcal{S}(\mathbb{R})$ and every $t > 0$, if $L^2 K N^{\gamma-1} \to 0$ as $N \to +\infty$, then*

$$\lim_{N\to\infty} \mathbb{E}^{\gamma}_{\nu_\rho}\Big[\Big(\int_0^t \frac{N^\gamma}{\sqrt{N}}\sum_{l\geq 1} H\Big(\frac{z_l}{N}\Big) V_{2,l,g}\big(\eta_s^N\big)ds\Big)^2\Big] = 0.$$

Proof Following the proof of Lemma 5.1, the expectation above becomes bounded by

$$Ct\sum_{l\geq 1}\sup_{h\in L^2(\nu_\rho)}\Big\{2\int \frac{N^\gamma}{\sqrt{N}} H\Big(\frac{z_l}{N}\Big) V_{2,l,g}(\eta)h(\eta)\nu_\rho(d\eta) - N^{1+\gamma}\langle h, -\mathcal{L}^S_{I_l^2} h\rangle_\rho\Big\}.$$

Using an appropriate A_l and the spectral gap inequality, last expression is bounded by

$$Ct\sum_{l\geq 1}\frac{N^\gamma}{N^2} H^2\Big(\frac{z_l}{N}\Big)(M+1)^2\,\mathrm{Var}(V_{2,l,g}, \nu_\rho).$$

Now, the proof ends as long as $\mathrm{Var}(V_{2,l,g}, \nu_\rho) \leq LC$, which is proved below. □

Remark 5.2 Here we show that $\mathrm{Var}(V_{2,l,g}, \nu_\rho) \leq LC$.

Since $\mathrm{Var}(V_{2,l,g}, \nu_\rho) \leq E_{\nu_\rho}[(V_{2,l,g})^2]$ and by the definition of $V_{2,l,g}$ we have that

$$\mathrm{Var}(V_{2,l,g}, \nu_\rho) \leq E_{\nu_\rho}\Big[\Big(\sum_{j\in I_l^2} E_{\nu_\rho}\Big[\sum_{x\in I_j} V_g(\eta(x))\Big| M_j\Big] - E_{\nu_\rho}\Big[\sum_{x\in I_l^2} V_g(\eta(x))\Big| M_l^2\Big]\Big)^2\Big].$$

By the definition of $V_g(\eta(x))$ and the notation introduced above, one can write the right hand side of last expression as

$$E_{\nu_\rho}\Big[\Big(\sum_{j\in I_l^2}\big(K\phi_j(\rho) - K\phi(\rho) - \phi'(\rho)K(\eta_j^K - \rho)\big) - M\phi_l(\rho) - M\phi(\rho) \\ - M\phi'(\rho)(\eta_l^M - \rho)\Big)^2\Big],$$

where $\phi_l(\rho)=E_{\nu_\rho}[g(\eta(0))|M_l]$ and $\eta_l^M=\frac{1}{M}\sum_{x\in I_l^2}\eta(x)$. Last expression can be written as

$$E_{\nu_\rho}\bigg[\bigg(M\bigg\{\frac{1}{M}\sum_{j\in I_l^2}\big(K\phi_j(\rho)-K\phi(\rho)-\phi'(\rho)K(\eta_j^K-\rho)\big)-\phi_l(\rho)$$

$$-\phi(\rho)-\phi'(\rho)(\eta_l^M-\rho)\bigg\}\bigg)^2\bigg]$$

$$=E_{\nu_\rho}\bigg[\bigg(M\bigg\{\frac{1}{L}\sum_{j\in I_l^2}\big(\phi_j(\rho)-\phi'(\rho)(\eta_j^K-\rho)-\phi_l(\rho)-\phi'(\rho)(\eta_l^M-\rho)\big)\bigg\}\bigg)^2\bigg]$$

$$=\frac{M^2}{L}E_{\nu_\rho}\bigg[\bigg(\frac{1}{\sqrt{L}}\sum_{j\in I_l^2}\big(\phi_j(\rho)-\phi'(\rho)(\eta_j^K-\rho)-\phi_l(\rho)-\phi'(\rho)(\eta_l^M-\rho)\big)\bigg)^2\bigg].$$

By the independence of the random variables $(\eta(x))_x$ under ν_ρ and the Central Limit Theorem, last expectation is of order

$$E_{\nu_\rho}\big[\big(\phi_j(\rho)-\phi(\rho)-\phi'(\rho)(\eta_j^K-\rho)\big)^2\big],$$

which we can bound from above by

$$2E_{\nu_\rho}\big[\big(\phi_j(\rho)-\phi(\eta_j^K)\big)^2\big]+2E_{\nu_\rho}\big[\big(\phi(\eta_j^K)-\phi(\rho)-\phi'(\rho)(\eta_j^K-\rho)\big)^2\big].$$

By the equivalence of ensembles the expectation on the left hand side of last expression is bounded by CK^{-2}. For the other, use Taylor expansion to have that

$$E_{\nu_\rho}\big[\big(\phi(\eta_j^K)-\phi(\rho)-\phi'(\rho)[\eta_j^K-\rho]\big)^2\big]\sim E_{\nu_\rho}\big[\big(\eta_j^K-\rho\big)^4\big]=O(K^{-2})$$

which is enough to finish the proof of the remark.

At this stage the restrictions on the size M of the second interval come from the four previous estimates. Since from the first stage of the argument we are able to take $K\sim O(N^{\frac{1-\gamma}{2}-\epsilon})$, then by the two previous estimates we can take $L\sim O(N^{\frac{1-\gamma}{4}})$. Collecting these facts together it implies that $M\sim O(N^{\frac{1-\gamma}{2}+\frac{1-\gamma}{4}-\epsilon})$.

Following the same arguments as before, take n sufficiently big for which in the n-th stage of the proof we have intervals, denoted by $\{I_m^n, m\geq 1\geq\}$ of length $K_n\sim N^{1-\gamma}$. At this stage, it remains to bound:

$$\mathbb{E}_{\nu_\rho}^\gamma\bigg[\bigg(\int_0^t\frac{N^\gamma}{\sqrt{N}}\sum_{m\geq 1}H\Big(\frac{z_m}{N}\Big)E_{\nu_\rho}\Big(\sum_{x\in I_m^n}V_g(\eta_s^N(x))\Big|M_m^n\Big)ds\bigg)^2\bigg],$$

where for each m, z_m is one point of the interval I_m^n and $M_m^n=\sigma(\sum_{x\in I_m^n}\eta(x))$. By Schwarz inequality and since ν_ρ is an invariant product measure, last expectation can be bounded by

$$t^2\frac{N^{2\gamma}}{N}\sum_{m\geq 1}\Big(H\Big(\frac{z_m}{N}\Big)\Big)^2E_{\nu_\rho}\bigg[\bigg(E_{\nu_\rho}\Big[\sum_{x\in I_m^n}V_g(\eta(x))\Big|M_m^n\Big]\bigg)^2\bigg].$$

Now, if one shows that the expectation above is of order $O(1)$, it implies the whole expression to be bounded from above by $\frac{N^{2\gamma}}{K_n}$ and since $\gamma < 1/3$, it vanishes as $N \to +\infty$. This is the only part in the proof that we impose the restriction on speed of the time scale.

Remark 5.3 Here we show that $E_{\nu_\rho}[(E_{\nu_\rho}[\sum_{x\in I_m^n} V_g(\eta(x))|M_m^n])^2] = O(1)$.

By the definition of $V_g(\eta(x))$, the expectation above is equal to

$$E_{\nu_\rho}\left[\left(E_{\nu_\rho}\left[\sum_{x\in I_m^n}\left(\phi_{K_n}(\rho)-\phi(\rho)-\phi'(\rho)(\eta_n^{K_n}-\rho)\right)\right]\right)^2\right]$$

and bounded from above by

$$2E_{\nu_\rho}\left[\left(E_{\nu_\rho}\left[\sum_{x\in I_m^n}\left(\phi_{K_n}(\rho)-\phi(\eta_n^{K_n})\right)\right]\right)^2\right]$$
$$+\,2E_{\nu_\rho}\left[\left(E_{\nu_\rho}\left[\sum_{x\in I_m^n}\left(\phi(\eta_n^{K_n})-\phi(\rho)-\phi'(\rho)(\eta_n^{K_n}-\rho)\right)\right]\right)^2\right],$$

where $\phi_{K_n}(\rho) = E_{\nu_\rho}[g(\eta(0))|M_m^n]$ and $\eta_n^{K_n} = \frac{1}{K_n}\sum_{x\in I_m^n}\eta(x)$. Now, the result follows if one applies the equivalence of ensembles to the expectation on the left hand side and Taylor expansion to the expectation on the right hand side of last expression, as explained above.

Remark 5.4 Here we give an application of the Boltzmann-Gibbs Principle for a linear functional associated to the one-dimensional Symmetric Zero-Range process evolving on the diffusive scaling N^2. Consider the Markov process η_t evolving on the parabolic time scale and with generator $\mathcal{L}^S$ given on local functions $f:\mathbb{N}^{\mathbb{Z}}\to\mathbb{R}$ by

$$\mathcal{L}^S f(\eta) = \sum_{\substack{x,y\in\mathbb{Z}\\|x-y|=1}} \frac{1}{2} g(\eta(x))[f(\eta^{x,y})-f(\eta)],$$

with $\eta^{x,y}$ as defined in the proof of Lemma 5.1. Let $\mathbb{P}_{\nu_\rho}$ be the probability measure in $D(\mathbb{R}_+,\mathbb{N}^{\mathbb{Z}})$ induced by ν_ρ and $\eta_\cdot$ evolving on the time scale N^2, namely $\eta_t^N = \eta_{tN^2}$ and denote by $\mathbb{E}_{\nu_\rho}$ the expectation with respect to $\mathbb{P}_{\nu_\rho}$. If one repeats the same steps as in the proof of Theorem 2.4, one can show that:

Corollary 5.3 *Fix $\beta < 1/2$. Then, for every $t>0$ and $H\in\mathcal{S}(\mathbb{R})$*

$$\lim_{N\to\infty}\mathbb{E}_{\nu_\rho}\left[\left(N^\beta\int_0^t\frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}}H\left(\frac{x}{N}\right)V_g(\eta_s^N(x))ds\right)^2\right]=0,$$

where $V_g(\eta(x))$ was defined in (2.5).

So, in order to observe non-trivial fluctuations for this field one has to take $\beta \geq 1/2$. A very interesting problem is to establish the limit of this linear functional when $\beta = 1/2$. For the Symmetric Simple Exclusion process starting from the Bernoulli product measure μ_α, evolving on the diffusive scaling N^2, for $\beta = 1/2$ and $g(\eta(x)) = (\eta(x)-\alpha)(\eta(x+1)-\alpha)$ which defines the quadratic density field, it was proved in [1] that this functional converges in law to a non-Gaussian singular functional of an infinite-dimensional Ornstein-Uhlenbeck process. The case for the Zero-Range process is still open.

Acknowledgements The author wants to express her gratitude to "Fundação para a Ciência e Tecnologia" for the grant /SFRH/BPD/39991/2007, to CMAT from University of Minho for support and to "Fundação Calouste Gulbenkian" for the Prize: "Estímulo à investigação" of the research project "Hydrodynamic limit of particle systems".

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